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UNSTEADY HEAT CONDUCTION IN PLATES OF POLYGONAL SHAPE

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Abstract—This paper deals with an unsteady heat-conduction problem in a plate of regular polygonal shape using conformal mapping techniques. The classical approach to an exact solution of the Fourier heat equation is the separation of variables technique. For more complicated boundaries, e.g. a hexagonal plate, it is convenient to transform the given shape onto a unit circle where the boundary conditions can be identically satisfied. However, the transformed partial differential equation can only be satisfied approximately. Two "weighted-residuals" techniques are used to solve it, and solutions are obtained for several polygonal shapes. The method can be extended easily in the case of some doubly connected regions of technical importance, e.g. the graphite brick of a gas cooled nuclear reactor.

NOMENCLATURE

k, thermal conductivity;
T, temperature;
∇^2, a two dimensional Laplacian operator;
t, time;
γ, separation constant;
$\varkappa$, thermal diffusivity;
ρ, density of the material;
c, specific heat of the material;
Q, rate of heat generated.

INTRODUCTION

AN ANALYTICAL solution of a heat-conduction problem must satisfy the Fourier heat equation as well as the initial and boundary conditions specified. If the boundary of the domain is natural to one of the common coordinate systems for which the partial differential equation can be solved by the classical method of separation of variables, the solution may be expressed, in general, in terms of known functions. For more "exotic" boundaries the natural system of coordinates must first be determined. This is a troublesome but not intractable problem. Even with the natural coordinate system known, it is very probable that the method of separation of variables would not be applicable [1].

As shown in [2] and [3] it is advantageous to conformally transform the given shape on to a unit circle. Under the transformation, however, the governing differential equation becomes quite complicated so that an exact solution is unlikely. Galerkin's method and a collocation technique are used to solve the transformed equation and good agreement is obtained between the two criteria.

The method is also applicable when dealing with double connected domains of a complicated shape, such as are common in solid propellant rocket motors, the graphite brick of a gas cooled nuclear reactor, etc.

THEORY

Two dimensional heat flow in an isotropic solid is governed by the equation

$$\frac{\partial}{\partial x_1}\left(k\frac{\partial T}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(k\frac{\partial T}{\partial x_2}\right) + Q = \rho c\frac{\partial T}{\partial t} \qquad (1)$$

where, k, thermal conductivity;
ρ, density of the material;
c, specific heat of the material;
Q, rate of heat generated.

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If the conductivity is constant throughout the body and no heat is generated within the solid, equation (1) becomes

$$\varkappa \nabla^2 T = \frac{\partial T}{\partial t} \tag{2}$$

where, $\varkappa$, thermal diffusivity;
and ∇^2, two dimensional Laplacian operator.

By assuming

$$T = T_1(x_1, x_2)\,\tau(t)$$

equation (2) yields:

$$\frac{1}{T_1}\nabla^2 T_1 = \frac{1}{\varkappa}\frac{\tau'}{\tau} = -\gamma. \tag{3}$$

Thus

$$\nabla^2 T_1 + \gamma T_1 = 0 \tag{4}$$

and

$$\tau' + \varkappa\gamma\tau = 0. \tag{5}$$

The solution of equation (5) is simply

$$\tau \sim \exp(-\varkappa\gamma t). \tag{6}$$

If the given domain is bounded by a complicated curve it will be necessary to use, in general, an approximate method like finite differences, point matching, finite elements, etc., to solve the differential system. The conformal mapping technique developed in [2] will be used in the present study. This method seems to yield smoother convergence than other well known methods as: finite differences, point matching, etc.

Equation (4) can be expressed in complex variable form as

$$4\frac{\partial^2 T_1}{\partial w\,\partial \bar{w}} + \gamma T_1 = 0 \tag{7}$$

where

$$w = R\,\mathrm{e}^{i\phi} = x_1 + ix_2. \tag{8}$$

The existence of a function $w = f(\xi)$ where $\xi = r\,\mathrm{e}^{i\theta}$ which maps the interior of a unit circle in the ξ-plane onto the interior of the arbitrary closed curve Γ in the w-plane is guaranteed by Riemann's theorem. Several approximate methods have been developed for the determination of the mapping function [5] and [6]. By changing variables, equation (7) is transformed to:

$$4\frac{\partial^2 T_1}{\partial \xi\,\partial \bar{\xi}} + \gamma \frac{\mathrm{d}w}{\mathrm{d}\xi}\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{\xi}} T_1 = 0. \tag{9}$$

where $\bar{\zeta}$ is the complex conjugate of ζ. Exact solution of (9) seems out of the question. Since the transformed region is now a unit circle let us form a sum of cylindrical harmonics as an approximate solution of equation (9):

$$T_1 \simeq \sum_{n=0}^{N}\sum_{m=1}^{M} A_{nm} J_n[\beta_{nm}\sqrt{(\xi\bar{\xi})}]\left(\frac{\xi}{\bar{\xi}}\right)^{n/2} \tag{10}$$

where J_n is the Bessel function of the first kind of order n and the β_{nm}'s are values to be determined by the homogeneous boundary conditions. Since an approximate solution is desired it will be profitable to simplify equation (10) even further.

For the case of an initial uniform temperature distribution T_0 and a homogeneous boundary condition

$$T(x_1, x_2, t) = 0 \qquad \text{on } \Gamma \tag{11}$$

it is reasonable to expect that isotherms in the ξ-plane will not depart drastically from circles. Consequently, the θ-dependence in equation (11) will be weak so that $n = 0$ terms will predominate. Equation (10) becomes

$$T_1 \simeq \sum_{m=1}^{M} A_{0m} J_0(\beta_{0m} r) \qquad r \leqslant 1. \tag{12}$$

The boundary condition in the ξ-plane is:

$$T(r, \theta, t)\big|_{r=1} = 0. \tag{13}$$

This allows for the evaluation of the β_{0m}'s since

$$J_0(\beta_{0m}) = 0$$

so, the β_{0m}'s turn out to be the roots of the Bessel function of first kind and order zero. Substitution from equation (12) into the transformed differential equation results in an error expression, $\varepsilon(r, \theta)$, which does not vanish in general since equation (12) is not an exact solution:

$$\varepsilon(r, \theta) = \sum_{m=1}^{M} A_{0m} f_m(r, \theta) J_0(\beta_{0m} r). \qquad (14)$$

Here,

$$f_m(r, \theta) = \gamma \left|\frac{dw}{d\xi}\right|^2 - \beta_{0m}^2 \qquad (15a)$$

$$\left|\frac{dw}{d\xi}\right|^2 = \frac{dw}{d\xi} \cdot \frac{d\bar{w}}{d\bar{\xi}}. \qquad (15b)$$

By using a suitable technique for minimizing the value of the "error function", $\varepsilon(r, \theta)$, over the unit circle, a system of M linear algebraic equations in the M unknown constants A_{0m}'s is obtained. Such a system can have a non-trivial solution only if the determinant of the coefficients of the unknown vanishes identically. From this determinantal equation, the separation constants γ can be obtained. The actual approximate solution of the problem is (considering only the terms corresponding to $n = 0$):

$$T \simeq \sum_{m=1}^{M} A_{0m} J_0(\beta_{0m} r) \exp(-\varkappa \gamma_m t) \qquad (16a)$$

where the A_{0m}'s are given by the Fourier–Bessel expansion:

$$A_{0m} = \frac{2T_0}{\beta_{0m} J_1(\beta_{0m})} = 1{\cdot}6019\, T_0; \quad -1{\cdot}0648\, T_0;$$
$$0{\cdot}8514\, T_0; \quad -0{\cdot}7295\, T_0; \quad +0{\cdot}6487\, T_0 \ldots. \qquad (16b)$$

For a doubly connected region like the one shown in Fig. 1 one can use a similar approach in the case of an initial uniform temperature distribution T_0 and homogeneous boundary conditions. It is advantageous now to transform the given region onto an annulus. The temperature distribution is then given by:

$$T \simeq \sum_{m=1}^{M} A_{0m} \left[J_0(\eta_{0m} r) - \frac{J_0(\eta_{0m} r_1)}{Y_0(\eta_{0m} r_1)} Y_0(\eta_{0m} r) \right] \exp(-\varkappa \gamma_m t). \qquad (17)$$

where r_1: inner radius of the annulus. It is convenient to take $r_1 = 1$. In equation (17) Y_0 is the Bessel function of second kind and zero order and the η_{0m}'s are the roots of the secular determinant

$$\begin{vmatrix} J_0(\eta_{0m} r_1) & Y_0(\eta_{0m} r_1) \\ J_0(\eta_{0m} r_2) & Y_0(\eta_{0m} r_2) \end{vmatrix} = 0. \qquad (18)$$

SOLUTION OF THE TRANSFORMED GOVERNING EQUATION

The general approach in the method of weighted residuals is to assume a trial function

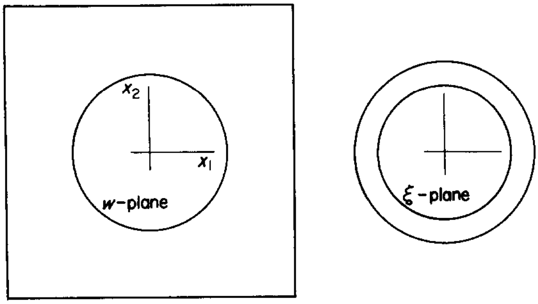

FIG. 1. Cross section of a graphite brick of a gas cooled nuclear reactor and its image in the ξ-plane.

with arbitrary constants; these are found by requiring that the trial solution satisfy the differential equation in some specified approximate sense. Two different criteria are used in the present study: collocation along arcs and Galerkin's method.

The criteria of collocation along arcs requires that the "error function", $\varepsilon(r, \theta)$, integrated along an arc $r = r_i$ be equal to zero for M values of r_i:

$$\int_0^{\theta_0} \varepsilon(r_i, \theta)\, r_i\, d\theta = 0 \qquad (i = 1, 2, 3, \ldots M) \qquad (19)$$

where θ_0 depends upon the number of axes of symmetry of the configuration (for a square peg with a concentric circular hole, $\theta_0 = \pi/4$; for a pentagon $\theta = \pi/5$, etc.). A homogeneous system of equations is obtained in the unknown coefficients. For a non-trivial solution the determinant of the coefficients of the unknowns must vanish. Thus an $(M \times M)$ determinantal equation is obtained in the separation constants

γ_m's. Using this collocation scheme one is able to use the expression defined by equation (12) without great computational effort.

It would be extremely complicated to use trial functions involving Bessel functions when using Galerkin's method, since this method requires that the error function $\varepsilon(r, \theta)$ be orthogonal with respect to each coordinate function, i.e.

$$\int_0^1\int_0^{\theta_0} \sum_{m=1}^{M} A_{0m}\left[-\beta_{0m}^2 + \gamma_m\left|\frac{\mathrm{d}w}{\mathrm{d}\xi}\right|^2\right] J_0(\beta_{0m}r)\,J_0(\beta_{0j}r)\,r\,\mathrm{d}r\,\mathrm{d}\theta = 0 \quad (j = 1, 2, 3, \ldots M).$$

It is more convenient, therefore, to use a simpler type of expansion when using Galerkin's method, i.e.

$$T_1(r, \theta) \simeq \sum_{m=1}^{M} A_{0m}\left[1 - (\xi\bar{\xi})^m\right]. \qquad (20)$$

The expression for the "error function" becomes

$$\varepsilon(r, \theta) = \sum_{m=1}^{M} A_{0m}\left[-4m^2 r^{2(m-1)} + \gamma_m\left|\frac{\mathrm{d}w}{\mathrm{d}\xi}\right|^2 (1 - r^{2m})\right]. \qquad (21)$$

By requiring

$$\int_0^{\theta_0}\int_0^1 \varepsilon(r, \theta)(1 - r^{2j})\,r\,\mathrm{d}r\,\mathrm{d}\theta = 0 \quad (j = 1, 2, 3, \ldots M) \qquad (22)$$

a determinantal equation in the γ_m's is obtained.

In the case of a doubly connected region it is convenient to use an expression of the form

$$T_1(r, \theta) \simeq \sum_{m=1}^{M} A_{0m}(r^{2m} - r_1^{2m})(r^{2m} - r_2^{2m}). \qquad (23)$$

APPLICATIONS

The method previously discussed will be now used in the determination of the temperature distribution in several plates of regular polygonal shape. The approximate mapping function which will be used in the calculation of the γ's is of the form

$$w = a \sum_{j=1}^{Q} a_{1+jp}\,\xi^{1+jp} \qquad (24)$$

where a is the apothem of the polygon; and p is the number of axes of symmetry. The coefficients have been calculated by the method discussed in [7] and are shown in Table 1.

By using equations (12), (14), (19) and (24) and taking $M = 5$, one obtains five values of the separation constant γ by the technique of collocation along arcs. These values, as well as the results obtained by Galerkin's method are shown in Table 2. Results for a square, pentagon, hexagon, heptagon and octagon are presented.

There is very good agreement between exact values and those obtained by the collocation scheme in the case of a square configuration. Galerkin's method yields extremely high values for the last three roots. The collocation scheme

Table 1. Mapping function coefficients

Polygon		Coefficients									
		a_1	a_{1+p}	a_{1+2p}	a_{1+3p}	a_{1+4p}	a_{1+5p}	a_{1+6p}	a_{1+7p}	a_{1+8p}	a_{1+9p}
Square*	$p = 4$	1·0800	−0·1080	0·0450	−0·0260	0·0174	−0·0127	0·0997	−0·0078	0·0064	−0·0054
Pentagon	$p = 5$	1·0515	−0·0683	0·0256	−0·0140	0·0091	−0·0068	0·0057	−0·0054	0·0069	−0·0041
Hexagon	$p = 6$	1·0376	−0·0484	0·0174	−0·0100	0·0091	−0·0047				
Heptagon	$p = 7$	1·0279	−0·0351	0·0112	−0·0055	0·0036	−0·0015				
Octagon	$p = 8$	1·0220	−0·0274	0·0086	−0·0046	0·0030	−0·0022	0·0017	−0·0012	0·0013	−0·0007

* The first ten terms of the exact mapping function for the square (using the Schwartz–Christoffel transformation).

Table 2. Comparison of values of $(\gamma_m a^2)$

Shape	Collocation along arcs*	Galerkin's method	Exact values
Square	4·933	4.935	4·935
	25·716	26·793	24·674
	63·069		64·152
	118·00		123·371
	190·99		202·326
Pentagon	5·221	5·221	
	27·405	26·937	
	67·273		
	125·28		
	201·55		
Hexagon	5·368	5·373	
	28·238	27·854	
	69·356		
	128·94		
	207·04		
Heptagon	5·471	5·471	
	28·805	34·798	
	70·778		
	131·49		
	210·97		
Octagon	5·537	5·537	
	29·160	35·046	
	71·656		
	133·08		
	213·42		

* $r_i = 0; 0{\cdot}2; 0{\cdot}4; 0{\cdot}6$ and $0{\cdot}8$

Table 3. T/T_0 *at the center of the square*
Comparison of results

Time (h)	Exact solution	Authors' results
0·1	0·928	0·925
0·2	0·654	0·649
0·3	0·426	0·422
0·4	0·274	0·271
0·5	0·176	0·174
0·6	0·113	0·112
0·7	0·0724	0·0716
0·8	0·0464	0·0459
0·9	0·0298	0·0295
1·0	0·0191	0·0189

$a = 0{\cdot}707$ ft; $\varkappa = 0{\cdot}45$ ft^2/h

gives, instead, very reasonable results in all cases with a minimum amount of labor.

Substituting the values of γ shown in Table 2 in equation (16a) one obtains approximate expressions for the temperature distribution in the ξ-plane and since a one-to-one correspondence exists between the ξ- and the w-plane the temperature variation in the w-plane is immediately known. Figures 2–4 show T/T_0 as function of time at the center of the plate for regular polygons circumscribed in a circle of radius $R = 1$ ft. In the case of a square plate the agreement between the exact and the approximate solution is very good, as demonstrated by Table 3 (see also [2] and [3]).

Equation (24) also maps with good accuracy [5] a doubly connected region with an outer regular polygonal shape and an inner concentric circular perforation when the web fraction* is considerably smaller than one. The determination of the temperature field [equation (17)] is then a straightforward application of the method.

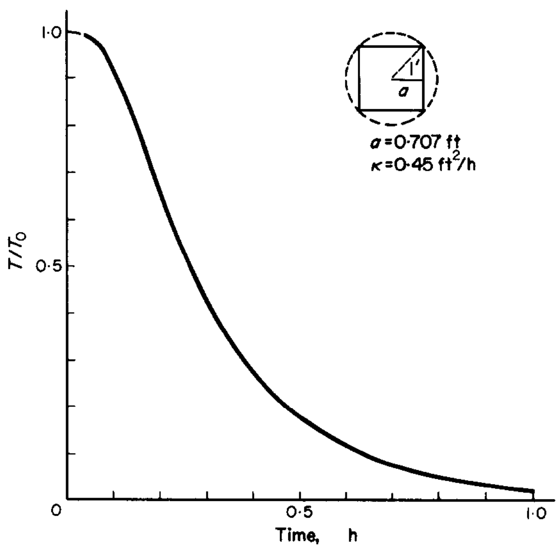

FIG. 2. Temperature variation with respect to time at the center of the square.

* The web fraction is defined as the ratio of the diameters of the circles circumscribing the inner and outer boundaries. If the web fraction is close to unity the determination of the mapping function is considerably more difficult since a system of coupled integral equations must be solved [6].

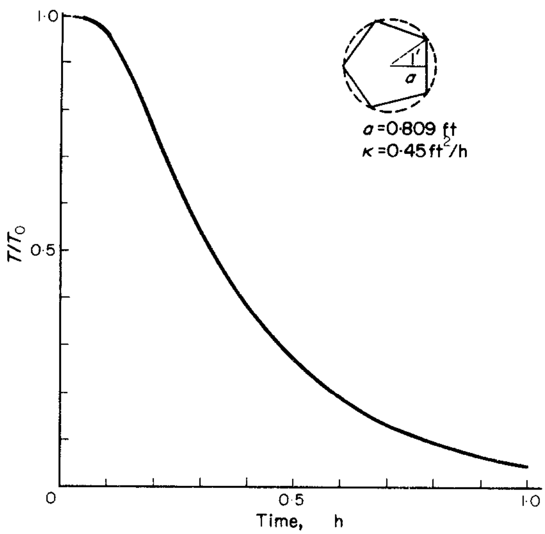

FIG. 3. Temperature variation with respect to time at the center of the pentagon.

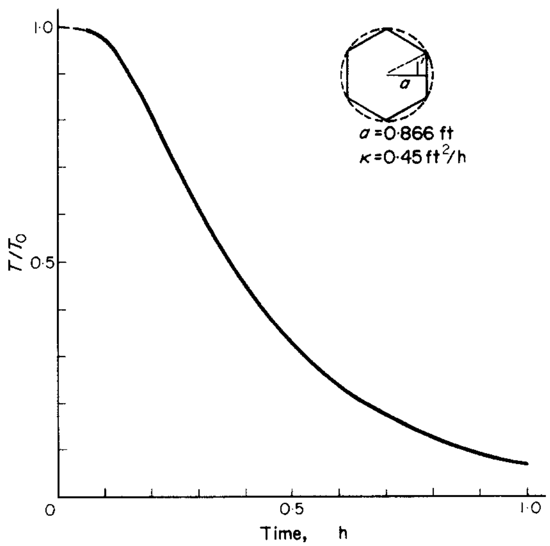

FIG. 4. Temperature variation with respect to time at the center of the hexagon.

CONCLUSIONS

An approximate solution of a class of unsteady-state heat-conduction problems has been obtained for plates of regular polygonal shape. The same approach can also be applied when dealing with some doubly connected domains of complicated shape such as the cross section of the graphite brick in a gas cooled nuclear reactor. The main advantage of the method is that, for the same initial and boundary conditions, the formal procedure and coordinate functions involved are the same for any boundary shape.

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Résumé—Cet article traite d'un problème de conduction transitoire de la chaleur dans une plaque en forme de polygone régulier en employant la technique de la représentation conforme. La méthode classique pour arriver à une solution de l'équation de la chaleur de Fourrier est la technique de la séparation des variables.

Pour des limites plus compliquées, par exemple une plaque hexagonale, il est commode de transformer la forme donnée en un cercle unitaire où les conditions aux limites peuvent être satisfaites identiquement. Cependant, l'équation aux dérivées partielles transformée peut seulement être satisfaite d'une façon approchée. On emploie deux techniques de "résidus pondérés" pour la résoudre, et l'on obtient des solutions pour plusieurs formes polygonales. La méthode peut être étendue faciliment au cas de certaines formes à connexion double et importantes techniquement, par exemple, la brique en graphite d'un réacteur nucléaire refroidi par gaz.

Zusammenfassung—Diese Arbeit behandelt ein Problem der instationären Wärmeleitung in einer Platte mit regulärer polygonaler Berandung, wobei die Methode der konformen Abbildung Verwendung findet. Der klassische Weg zur exakten Lösung der Fourierschen Wärmeleitungsgleichung erfolgt über die Methode der Trennung der Variablen.

Für kompliziertere Berandungen, z.B. eine hexagonale Platte, empfiehlt es sich, die gegebene Berandung auf einen Einheitskreis abzubilden, wo die Randbedingungen identisch erfüllt werden können. Die transformierten partiellen Differentialgleichungen können allerdings nur näherungsweise erfüllt werden. Es werden zwei Methoden der "gewogenen Residuen" zu deren Integration herangezogen und Lösungen für mehrere polygonale Berandungen gewonnen.

Die Methode kann im Falle einiger zweifach zusammenhängender Bereiche leicht erweitert werden. Als Beispiel sei der Graphitziegel eines gasgekühlten Atomreaktors genannt.

Аннотация—В данной статье рассматривается задача о нестационарной теплопроводности правильной многоугольной пластины с помощью метода конформных отображений. Метод разделения переменных является классическим методом точного решения уравнения теплопроводности Фурье. Для более сложных конфигураций, например, шестиугольной пластины, удобно преобразовать её на внутренность единичного круга,, где идентично удовлетворяются граничные условия. Однако, можно только приближенно удовоетворить преобразованное диференциальное уравнение в частных производных. Для его решения используются два метода «взвешенных разностей». Получены решения для нескольких многоугольников. Можно легко применить этот метод для случая нескольких двусвязных областей, например, графитовый кирпич реактора с газовым охлаждением.